IIT-JEE-Mathematics-Mains–2000

MAINS

Time : two hours Max. Marks : 100

General Instructions :
1. There are ten questions in this paper. Attempt all Questions.
2. Answer each question starting on a new page. The corresponding question number must be written in the left margin. Answer all the parts of a question at one place only.
3. Use only Arabic numerals (0, 1, 2 …………9) in answering the questions irrespective of the language in which your answer.
4. Use of logarithmic tables is not permitted.
5. Use of calculator is not permitted.

PART- B

1. (a) The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms of it. Prove that resulting sum is the square of an integer. (4)

(b) For any positive integers m, n (with $n \geq m$), let $\binom{n}{m} = {}^nC_m$. Prove that

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \ldots\ldots + \binom{m}{m} = \binom{n+2}{m+2}$$

Hence or otherwise, prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \ldots\ldots + (n-3+1)\binom{m}{m} = \binom{n+2}{m+2}$$ (6)

2. (a) If α, β are the roots of $ax^2+bx+c=0$,(a?0) and α+δ, β+δ are the roots of $Ax^2+Bx+C=0$,(A?0)for some constant δ, then prove that
$(b^2-4ac)/a^2 =(B^2-4AC)/A^2$ (4)

(b) For every positive integer, prove that
?(4n+1) < ?n + ?(n+1) < ?(4n+2)
Hence or otherwise, prove that [?n+?(n+1) = [?(4n+1]), where [x] denotes the greatest integer not exceeding x. (6)

3. (a) In any triangle ABC, prove that
Cot A/b + cot B/2 + cot C/2 = A/2 cot B/2 cot C/2 (3)

(b) Let ABC be a triangle with incentre I and inradius r. Let D, E, F be the feet of the perpendiculars from I to the sides BC, CA and AB respectively. If r_1, r_2 and r_3 are the radii of circles inscribed in the quadrilaterals AFIE, BDIF and CEID respectively, prove that $r_1/(r - r_1) + r_2/(r - r_2) + r_3/(r - r_3) = (r_1 r_2 r_3)/(r - r_1)(r-r_2)(r-r_3)$

4. For points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ of the coordinate plane, a new distance d(P, Q) is defined by d (P, Q) = ? $x_1–x_2$? +? $y_1–y_2$? . Let O = (0, 0) and A=(3, 2). Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from O and A consists of the union of line segment of finite length and an infinite ray. Sketch this set in a labelled diagram.

5.

(a) Prove that for all values of θ

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0.$$

(b) Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B, C to the major axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$, (a>b) meets the ellipse respectively at P, Q, R so that P, Q, R lie on the same side of the major axis as A, B, C respectively. Prove that the normals to the ellipse drawn at the points P, Q and R are concurrent.

6. Let C_1 and C_2 be, respectively, the parabolas $x^2 = y – 1$ and $y^2 = a^2$. Let P be any point on C_1 and Q be any point on C_2. Let P_1 and Q_1 be the reflections of P and Q, respectively, with respect to the line y = x. Prove that P_1 lies on C_2, Q_1 lies on C_1 and PQ ? min {PP_1, QQ_1}. Hence or otherwise, determine points P_0 and Q_0 on the parabolas C_1 and C_2 respectively such that $P_0 Q_0$? P_0 for all pairs of points (P, Q) with P on C_1 and Q on C_2.

7.

(a) Suppose $P(x) = a_0 + a_1 x + a_2 x^2 + \dots\dots\dots + a_n x^n$. If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$, prove that $|a_1 + 2a_2 + \dots\dots + na_n| \le 1$

(b) For x > 0, let $f(x) = \int_1^x \frac{\ln t}{1+t}\,dt$. Find the function $f(x) + f(1/x)$ and show that $f(e) + f(1/e) = 1/2$. Here $\ln t = \log_e t$.

8. A country has food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after n years, where n is the smallest integer bigger than
or equal to **(In 10 - In 9) / (In (1.04)- 0.03).**

9. (a) A coin has probability p of showing head when tossed. It is tossed n times. Let pn, denote the probability that no two (or more) consecutive heads occur. Prove that $p_1 = 1$, $p_2 = 1 – p^2$ and $p_n = (1 – p). p_n–1 + p (1 – p) p_n – 2$ for all n ? 3.

(b) In (a), prove by induction on n, that p_n = A αn + Bβn for all n ? 1, where α and β are the roots of the quadratic $x^2 - (1 - p)x - p(1 - p) = 0$ and

$$A = \frac{p^2+\beta-1}{\alpha\beta-\alpha^2}, B = \frac{p^2+\alpha-1}{\alpha\beta-\beta^2}.$$

10. Let ABC and PQR be any two triangles in the same plane. Assume that the perpendiculars from the points A, B, C to the sides QR, RP, PQ respectively are concurrent. Using vector methods or otherwise, prove that the perpendiculars from P, Q, R to BC, CA, AB respectively are also concurrent.